

POINTWISE BOUNDED APPROXIMATION AND ANALYTIC CAPACITY OF OPEN SETS

BY

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ABSTRACT. We examine the semi-additivity question for analytic capacity by studying the relation between the capacities of bounded open sets and their closures.

Introduction. Definitions and basic properties of analytic capacity, along with its most important uses, can be found in [4] or [20]. Let \mathbb{C} denote the complex plane, and \mathbb{C}^* the Riemann sphere. We write $\gamma(S)$ and $\alpha(S)$ for the analytic capacity and continuous analytic capacity of a set $S \subset \mathbb{C}$. The class of functions defining $\gamma(S)$ will be called admissible, and the class defining $\alpha(S)$ will be called C -admissible. If f satisfies the defining conditions for admissibility, except possibly that $f(\infty) \neq 0$, then f will be called "admissible" (and the quotation marks are part of the word). An Ahlfors function for a compact set K is a function f admissible for K , with $f'(\infty) = (2\pi i)^{-1} \int f(z) dz = \gamma(K)$, where the path of integration is any circle surrounding K . Such a function always exists, and is uniquely determined on the unbounded component of $\mathbb{C}^* - K$.

The word neighborhood will always mean an open set. We will always use the letter D for an open disc; $D(z, \delta)$ will be the open disc with center z and radius δ ; and if $r > 0$, then rD will be the open disc with the same center as D , and with radius equal to r times the radius of D . The symbol dm will refer to integration with respect to planar Lebesgue measure, and we will use a double integral symbol for such integration. In place of the differential operator $\frac{1}{2}(\partial/\partial x + i\partial/\partial y) = \partial/\partial \bar{z}$ we will use the notation of the gradient. When the symbol, $\text{grad } \phi$, appears in an integral, it is understood to mean $(\text{grad } \phi)(w)$, where w is the variable of integration. If $K \subset \mathbb{C}$ is compact, then $C(K)$ is the Banach algebra of all continuous, complex valued functions on K , with the supremum norm. $R(K)$ is the closed subalgebra generated by all rational functions with poles off K . If U is open, $H^\infty(U)$ is the Banach algebra of all bounded analytic functions on U , with the supremum norm.

It has been conjectured that capacity has the property of semi-additivity:

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Does there exist a constant M with the property that $\alpha(K \cup L) \leq M(\alpha(K) + \alpha(L))$ and $\gamma(K \cup L) \leq M(\gamma(K) + \gamma(L))$ whenever K and L are compact disjoint sets? Semi-additivity is known to be true in certain special cases, and it is even true that $\gamma(K \cup L) \rightarrow \gamma(K) + \gamma(L)$ as the distance between K and L approaches infinity. See Satz 4 of [12]. Studying the semi-additivity question in detail in [3], A. Davie has found a number of equivalent formulations. In his Theorem 5.11, Davie shows that semi-additivity of α is equivalent to that of γ , and also that both of these are equivalent to the following statement. If E is compact and $\gamma(E) = 0$, then $\gamma(E \cup F) = \gamma(F)$ for all sets F . We claim that it suffices here to take F to be an open set.

PROPOSITION. *The relation $\gamma(E \cup F) = \gamma(F)$ is true for all sets F and for all compact sets E with $\gamma(E) = 0$ iff the relation $\gamma(E \cup V) = \gamma(V)$ is true for all bounded open sets V and all compact sets E with $\gamma(E) = 0$.*

PROOF. We begin by noting the fact that if B and C are compact sets, then $B - C$ is locally compact. Now suppose $\gamma(E \cup V) = \gamma(V)$ for some compact set E and for all bounded open sets V . Let F be any set, and let $K \subset E \cup F$ be compact. Then $K - E$ is locally compact, and also $K - E \subset F$. In §1 of [3], Davie defines a set function γ^* by $\gamma^*(S) = \inf\{\gamma(U) : U \text{ is open, } S \subset U\}$, and in Theorem 5.1 of the same paper, he shows that there is an absolute constant A satisfying $\gamma^*(S) \leq A\gamma(S)$ whenever S is locally compact. Therefore we can choose an open set U with $K - E \subset U$ such that $\gamma(U) \leq A\gamma(K - E) + \epsilon$. If we intersect U with an open disc D which contains K , then we get a bounded open set V such that $K - E \subset V$ and $\gamma(V) \leq \gamma(U) \leq A\gamma(K - E) + \epsilon$. Therefore $K \subset E \cup V$, and $\gamma(K) \leq \gamma(E \cup V) = \gamma(V) \leq A\gamma(K - E) + \epsilon \leq A\gamma(F) + \epsilon$. Since ϵ and K are arbitrary, $\gamma(E \cup F) \leq A\gamma(F)$, and then $\alpha(E \cup F) \leq A\gamma(F)$.

The conclusion we have established is part two of Theorem 5.11 of [3]. As that is equivalent to part three, we have proved half of our proposition. The other half is trivial.

In attempting to prove a statement like $\gamma(E \cup V) = \gamma(V)$ we see that the only difficulty is with the portion of E which belongs to ∂V . Compact sets of zero analytic capacity are sets of removable singularities for bounded analytic functions. A function admissible for a compact subset K of $E \cup V$ will therefore extend analytically across every point of $K - \bar{V}$ to become admissible for $K \cap \bar{V} \subset V \cup (E \cap \partial V)$. To prove $\gamma(V) = \gamma(E \cup V)$ it thus suffices to prove $\gamma(V) = \gamma(V \cup (E \cap \partial V))$. Note that $E \cap \partial V$ is a compact set of zero analytic capacity. (Incidentally, the remarks of this paragraph show that $\gamma(K \cup L) = \gamma(L)$ whenever K and L are compact and $\gamma(K) = 0$.)

This leads to the question of whether or not it is possible to add part of ∂V on to V without changing the analytic capacity. We plan to show that in many cases *all* of ∂V can be added on, thus yielding the relation $\gamma(V) = \gamma(\bar{V})$.

In the general case, a certain portion of ∂V can always be added on. Finally we will show that the relation $\gamma(V) = \gamma(\bar{V})$ being true for a certain class of bounded open sets is equivalent to the semi-additivity of analytic capacity.

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1. The main theorem. Our principal result about pointwise bounded approximation is Theorem 1 below. This is very similar to Vitushkin's Theorem and a large portion of the proof is the same. One way in which it differs is that we need the following difficult result of T. Gamelin and J. Garnett. This is the special case of Theorem 6.7 of [6] in which P is taken to be the empty set.

REDUCTION OF NORM THEOREM. *Let $U \subset \mathbb{C}^*$ be open and let $Q \subset \partial U$ be compact. Assume $R(Q) = C(Q)$. Let J be a uniformly closed subalgebra of $H^\infty(U)$. Suppose that for each $F \in J$, there is a uniformly bounded sequence $\{F_n\}_{n=1}^\infty$ in $H^\infty(U)$ such that each F_n extends to be analytic in a neighborhood of Q , and such that $\{F_n\}$ converges uniformly to F on each subset of U at positive distance from Q . Then there is a sequence in $H^\infty(U)$ having all these properties of the F_n , and in addition satisfying $\|F_n\| \leq \|F\|$.*

COROLLARY. *The conclusion of the Reduction of Norm Theorem remains valid if we replace the hypothesis " $Q \subset \partial U$ " by " $Q \subset \mathbb{C} - U$."*

It does not appear that this corollary follows from the statement of the Reduction of Norm Theorem. However the corollary does follow from the proof of that theorem if some minor changes are made. As a matter of fact, it is this modification of the Reduction of Norm Theorem, and not the theorem itself, which is used in the proof of Theorem 6.9 of [6].

THEOREM 1. *Let $W \subset \mathbb{C}^*$ be an open set containing the point ∞ . Let $E \subset \mathbb{C} - W$ be compact, with $R(E) = C(E)$. Then the first four statements below are equivalent, and each of them implies the fifth one. In the special case that E has Lebesgue measure zero, all the five statements are equivalent.*

- (a) $\gamma(G - W) = \gamma(G - (W \cup E))$ for every bounded open set G .
- (b) $\gamma(D - W) = \gamma(D - (W \cup E))$ for every open disc D .
- (c) For each $z \in E \cap \partial W$, there exists $r = r(z) \geq 1$ such that

$$\liminf_{\delta \rightarrow 0} \frac{\gamma(D(z, r\delta) - (W \cup E))}{\gamma(D(z, \delta) - W)} > 0.$$

- (d) There exist numbers $r \geq 1$, $m \geq 1$, $\delta_0 > 0$ with the following property.

For every $z \in \mathbb{C}$ and every number δ with $0 < \delta < \delta_0$,

$$(*) \quad \gamma(D(z, \delta) - W) \leq m\gamma(D(z, r\delta) - (W \cup E)).$$

(e) Given $f \in H^\infty(W)$ there is a sequence $\{F_n\}_{n=1}^\infty$ in $H^\infty(W)$ such that

- (i) F_n extends analytically to a neighborhood, depending on n , of E ;
- (ii) $|F_n| \leq \|f\|$ on W and on this neighborhood;
- (iii) $\{F_n\}$ converges to f uniformly on each subset of W at a positive distance from E .

PROOF. We will show that $d \Rightarrow e$ and $d \Rightarrow a \Rightarrow b \Rightarrow c \Rightarrow d$. If $m(E) = 0$ we conclude with the implication $e \Rightarrow d$. We wish to begin with the demonstrations that $d \Rightarrow e$ and $d \Rightarrow a$ since they will be very similar, and we will use Vitushkin's approximation procedure.

Assume d is true and let $f \in H^\infty(W)$. We extend f to \mathbb{C}^* by setting $f \equiv 0$ off W . Let $0 < \delta < \delta_0$ and form Vitushkin's partition of unity $\{\phi_k\}_{k=1}^\infty$ corresponding to the number δ . See §2 of [3] as a reference. (We will write D_k for Δ_k .) Let $\{D_k: k = 1, 2, \dots, N(\delta)\}$ be an enumeration of all discs of Vitushkin's construction which contain points within δ of E . We have $\phi \equiv \phi_1 + \dots + \phi_{N(\delta)} \equiv 1$ on the δ -neighborhood of E . Let $f_k = T_{\phi_k}f$ where we again use §2 of [3] as a reference. Then f_k is a bounded Borel measurable function on \mathbb{C}^* , which is analytic off a compact subset of $D_k - W$, and which is zero at ∞ . We have the bound $\|f_k\| \leq 160\|f\|$. Since $\phi \equiv 1$ on the δ -neighborhood of E , $f - \sum f_k$ is analytic there, which we may see by examining $T_\phi f$.

We wish to produce functions g_k which approximate the f_k , and for this we split the indices $1, 2, \dots, N(\delta)$ into two classes, depending on whether or not $\gamma(rD_k - (W \cup E))$ is zero. If this is zero, then $(*)$ says that $\gamma(D_k - W) = 0$. Since f_k is bounded and analytic off a compact subset of $D_k - W$, f_k extends analytically to \mathbb{C}^* . (Compact sets of zero analytic capacity are removable singularities for bounded analytic functions.) Thus $f_k \equiv f_k(\infty) = 0$, and in this case we take $g_k = 0$.

On the other hand, suppose $\gamma(rD_k - (W \cup E)) \neq 0$. If one defines analytic diameters and analytic centers as in §VIII.6 of [4], then it is possible to proceed as on pp. 101–102 of [20] to produce bounded measurable functions g_k , analytic off compact subsets of the $rD_k - (W \cup E)$, such that $\|g_k\| \leq M(m, r)\|f\|$, and such that $g_k - f_k$ has a triple zero at ∞ . Here, $M(m, r)$ is a constant depending only on m and r , and if we assume that $r \geq 2$ is an integer, we may take $M(m, r) = \text{absolute constant} \cdot mr^5$. This assumption in no way harms the statement of the theorem. We write $M_1 = M(m, r)$.

If z_k is the center of D_k , we have $|z - z_k|^3 |f_k(z) - g_k(z)| \leq r^3 \delta^3 (160 + M_1)\|f\|$ on the boundary of rD_k . This inequality is valid off rD_k by the Maximum Modulus Principle, and is also true on rD_k . Thus it holds on

C^* . We will use this to estimate $\sum |f_k - g_k|$.

Fix $z \in C$, and for each positive integer j , let C_j be the circle with center z and radius $j\delta$. Then each disc D_k meets at least one and at most two of the C_j , except that if z happens to be the center of one of the D_k , then that particular D_k will not meet any of the C_j . Let N_j be the number of discs D_k which meet C_j . From the construction of the D_k , an arbitrary disc of radius δ can meet at most 64 of the D_k . Since C_j can be covered by $10j$ discs of radius δ , each of which meets at most 64 of the D_k , we see that $N_j \leq 640j$. Recalling that at most one disc D_k fails to meet a circle C_j , we find

$$(1) \quad \sum_{k=1}^{N(\delta)} |f_k(z) - g_k(z)| \leq (160 + M_1) \|f\| \left[1 + N_1 + \sum_{j=2}^{\infty} N_j / (j-1)^3 \right] \leq M_2 \|f\|,$$

since $|z - z_k| \geq r(j-1)\delta$ whenever $j \geq 2$ and D_k meets C_j .

Now let us suppose that $\text{dist}(z, E) > d > 0$. Then if δ is chosen less than $d/4r$ we have $\text{dist}(z, rD_k) > \frac{1}{2}d$ for $k = 1, 2, \dots, N(\delta)$. Then $j \leq d/4r\delta$ implies $N_j = 0$, while $j > d/4r\delta > 1$ implies $j \geq 2$, which in turn implies that $1/(j-1) \leq 2/j$, and (1) becomes

$$\begin{aligned} \sum_{k=1}^{N(\delta)} |f_k(z) - g_k(z)| &\leq (160 + M_1) \|f\| \sum_{j > d/4r\delta} 8N_j / j^3 \\ (2) \quad &\leq (160 + M_1) \|f\| \cdot 8(4r\delta/d)^3 \sum_{j > d/4r\delta} N_j \\ &\leq M_3 \|f\| (\delta/d)^3 \sum_{j=1}^{\infty} N_j \leq M_3 \|f\| (\delta/d)^3 \cdot 2N(\delta) \end{aligned}$$

since each disc D_k meets at most two of the C_j . We recall that $N(\delta)$ is the number of discs with which we originally covered E . Each of these discs contains points within δ of E , and so the union of these discs lies inside some disc D whose radius is $\text{diam } E + 6\delta$. By the construction of the D_k , each point of D is covered at most 25 times by $\bigcup D_k$. Therefore $N(\delta) \cdot \pi\delta^2 \leq 25\pi(\text{diam } E + 6\delta)^2$, and inequality (2) becomes

$$(3) \quad \sum_{k=1}^{N(\delta)} |f_k(z) - g_k(z)| \leq M_4 \|f\| d^{-3} \cdot \delta (\text{diam } E + 6\delta)^2.$$

Now we define $F_\delta = f - \sum_{k=1}^{N(\delta)} (f_k - g_k)$. This function is analytic in W and in a neighborhood of E (because $f - \sum f_k$ is analytic in a neighborhood of E). It is bounded on C , hence on C^* , by $(1 + M_2) \|f\|$, where $M_2 = M_2(m, r)$. Finally $F_\delta \rightarrow f$ uniformly in z , as $\delta \rightarrow 0$, provided z remains at a positive distance (at least d) from E . This is almost the result we are after, and that result follows from this one and the Reduction of Norm Theorem. We remark that the compactness of E and the

the hypothesis $R(E) = C(E)$ are used only at this last step. This proves the implication $d \Rightarrow e$.

The implication $d \Rightarrow a$ is obtained in the same way, so we will only point out the differences. Again we assume that d is true and that $r \geq 2$. Let G be a bounded open set, let $\epsilon > 0$, let $K \subset G - W$ be compact with $\gamma(K) \geq \gamma(G - W) - \epsilon$, and let f be an arbitrary element of $H^\infty(C^* - K)$. Since the compact set K is contained in the open set G , there is a number η with $0 < \eta < \text{dist}(K, C - G)$. Let $0 < \delta < \min\{\delta_0, \eta/4r\}$. Now we proceed as above. Set $f \equiv 0$ on K . Form the partition of unity corresponding to δ , and let $\{D_k: k = 1, 2, \dots, N(\delta)\}$ be the discs which satisfy $\text{dist}(D_k, E \cap K) < \delta$. Define f_k as before. Since $\Sigma \phi_k \equiv 1$ on the δ -neighborhood of $E \cap K$, $f - \Sigma f_k$ is analytic there, and since each f_k is analytic on the neighborhood $C^* - K$ of $E - K$, $f - \Sigma f_k$ is actually analytic on a neighborhood of E .

Define g_k as before. Each function g_k is analytic off a compact subset of $rD_k - (W \cup E)$, but that is not enough. We need to know that g_k is analytic off a compact subset of G , and we show now that this is true. Let z_k be the center of D_k . Since D_k contains points within δ of $E \cap K$, let $w_k \in D_k$ and $e_k \in E \cap K$ satisfy $|w_k - e_k| < \delta$. Then for each point z belonging to rD_k we have $|z - e_k| \leq |z - z_k| + |z_k - w_k| + |w_k - e_k| \leq r\delta + \delta + \delta = (r + 2)\delta \leq 2r\delta \leq \frac{1}{2}\eta$. So every point of every disc rD_k lies within $\frac{1}{2}\eta$ of $E \cap K$, and therefore lies inside G . If we write N for the $\frac{1}{2}\eta$ -neighborhood of K , then g_k is analytic off a compact subset of $\bar{N} - (W \cup E) \subset G - (W \cup E)$.

Therefore the function F_δ is analytic on W , on a neighborhood of E , and off the compact subset \bar{N} of G . This function is bounded on C^* by $(1 + M_2)\|f\|$, and if we take a sequence $\delta_n \searrow 0$, then the corresponding functions $F_n = F_{\delta_n}$ converge uniformly to f on any set at positive distance from $E \cap K$. Let U be the open set $W \cup (C^* - \bar{N})$ and let Q be the compact set $E \cap \bar{N}$. We have $Q \subset C - U$, and since $R(E) = C(E)$ we also have $R(Q) = C(Q)$ by Tietze's Extension Theorem. Let $J = H^\infty(C^* - K)$, a closed subalgebra of $H^\infty(U)$. The Reduction of Norm Theorem tells us that for each $f \in J$ we can find a sequence $\{F_n\}$ in $H^\infty(U)$ with F_n analytic on a neighborhood of Q , such that $\|F_n\| \leq \|f\|$, and $F_n \rightarrow f$ uniformly on any subset of U at positive distance from Q .

Let h be the Ahlfors function for K . Then the restriction of h to $C^* - K$ belongs to J , and we have a sequence $\{H_n\}$ of functions "admissible" for compact subsets of $\bar{N} - [W \cup (E \cap \bar{N})] = \bar{N} - [W \cup E] \subset G - [W \cup E]$. Choose $R > 0$ so large that $E \cup G \subset D(0, \frac{1}{2}R)$ and that the circle C with center 0 and radius R lies in W . Choose a positive integer n so that $\|H_n - h\|_C \leq \epsilon/R$. This yields $\gamma(G - W) \leq \gamma(K) + \epsilon = h'(\infty) + \epsilon \leq |H'_n(\infty)| + 2\epsilon \leq \gamma(G - (W \cup E)) + 2\epsilon$. Since ϵ is arbitrary, the proof of $d \Rightarrow a$ is complete.

The implications $a \Rightarrow b \Rightarrow c$ are trivial. To obtain c from b , take $r(z) = 1$

for every z . The proof of (the contrapositive of) the implication $c \Rightarrow d$ is so very similar to the corresponding portion of Vitushkin's Theorem, that we will omit it, except for one remark. The proof proceeds by repeated application of (the contrapositive of) the implication $d \Rightarrow a$, which we just established. This implication is applied to various open sets, and we must make sure that each of these open sets is a neighborhood of ∞ . We choose $R \geq 10$ so that $E \subset D(0, R)$ and $C^* - D(0, R) \subset W$. Then we add on $C^* - \bar{D}(0, 2R)$ to each of the open sets. This is a minor technical point which does no damage because everything takes place well inside $D(0, R)$.

The only implication remaining is that $e \Rightarrow d$ whenever $m(E) = 0$. This will be an imitation of Lemma 8.2 of [6]. Let $r > 1$ and $\delta_0 > 0$ be arbitrary and let $m = 240r/(r - 1)$. Let D be any open disc of radius δ . Let f be a function admissible for a compact subset of $D - W$. Since f is analytic on W , there is a sequence $\{f_n\}$ associated to f by the hypothesis e . Let W_n be a neighborhood of E on which f_n is analytic. Choose $R > 0$ so that $rD \subset D(0, R)$ and $C^* - D(0, R) \subset W$. Let $\epsilon > 0$ and let G be a neighborhood of E such that $m(G) \leq \pi \epsilon^2 \delta^2 (r - 1)^2 / 6400(4R^2)$. Redefine f_n so that $f_n \equiv 0$ off $W \cup (G \cap W_n)$. Since the sequence $\{f_n\}$ lies in the closed unit ball of $L^\infty(dm)$, some subsequence, which we will call $\{f_n\}$, converges weak star in $L^\infty(dm)$ to a limit function in the closed unit ball. We write this limit function as $f + h$, where $h \in L^\infty(dm)$. Then $\|h\|_\infty \leq 2$ and $h = 0$ a.e. on W .

Define $\tilde{h} \in L^\infty(dm)$ by $\tilde{h} \equiv 0$ on $W \cup G$ and $\tilde{h} \equiv h$ off $W \cup G$. Let ϕ be a continuously differentiable function on C^* , with $0 \leq \phi \leq 1$, $\phi \equiv 1$ on \bar{D} , $\phi \equiv 0$ off rD , and $\|\text{grad } \phi\| \leq 20/(r - 1)\delta$. Consider the functions $T_\phi f_n$ and the function $\tilde{H} = T_\phi \tilde{h} - \tilde{h}\phi$. The functions $T_\phi f_n$ are bounded Borel functions which are analytic off a compact subset of rD , analytic on $W \cup W_n$, and zero at ∞ . Each is bounded in modulus by $160r/(r - 1)$. The function \tilde{H} is the convolution of a bounded measurable function and a locally integrable function, and is continuous on C^* by the proof of Lemma 3.1.5 of [1]. From Theorem 10.7 of [14] we see that \tilde{H} is analytic on $C - [\text{supp } \phi - (W \cup G)]$. By Lemma 3.1.1 of [1], \tilde{H} is bounded by $40/\pi(r - 1)\delta \cdot 2\sqrt{\pi m(\text{supp } \phi)} < 80r/(r - 1)$, and so \tilde{H} is analytic on $C^* - [\text{supp } \phi - (W \cup G)] = W \cup G \cup (C^* - \text{supp } \phi)$. From the above bounds and the definition of capacity,

$$\begin{aligned} |(T_\phi f_n)'(\infty) - \tilde{H}'(\infty)| &\leq \frac{240r}{r - 1} \gamma(\text{supp } \phi - (W \cup (G \cap W_n))) \\ (4) \qquad \qquad \qquad &\leq \frac{240r}{r - 1} \gamma(rD - (W \cup E)). \end{aligned}$$

Recall the choice of R . Let C be the circle with center 0 and radius $2R$. If $z \in C$ then $\text{dist}(z, \text{supp } \phi) > R$ and therefore the function of w , $(w - z)^{-1} \cdot (\text{grad } \phi)$, belongs to $L^\infty(dm)$. If we split up the defining integrals as an integral

over $\text{supp } \phi - G$ and an integral over $\text{supp } \phi \cap G$, we see that

$$\begin{aligned} T_\phi f_n(z) - \tilde{H}(z) &\rightarrow \frac{1}{\pi} \iint \frac{f(w)}{w-z} \text{grad } \phi \, dm(w) \\ &+ \frac{1}{\pi} \iint_{\text{supp } \phi \cap G} \frac{h(w)}{w-z} \text{grad } \phi \, dm(w). \end{aligned}$$

The first integral here is immediately recognizable as $T_\phi f(z) - f(z)\phi(z) = T_\phi f(z)$ since $\phi(z) = 0$. This is actually equal to $f(z)$ as we now show. $T_\phi f - f$ is analytic on D because $\phi \equiv 1$ there. Since f is analytic off a compact subset of D , so is $T_\phi f$. Thus $T_\phi f - f$ is entire, hence constant by Liouville's Theorem. This constant is zero since that is the value of the function at ∞ . Therefore, if $z \in C$,

$$T_\phi f_n(z) - \tilde{H}(z) - f(z) \rightarrow \frac{1}{\pi} \iint_G \frac{h(w)}{w-z} \text{grad } \phi \, dm(w)$$

since the integral over G is the same as over $G \cap \text{supp } \phi$.

For each $z \in C$ there is a positive integer $N(z)$ with the property that $|T_\phi f_n(z) - \tilde{H}(z) - f(z)| \leq \epsilon/2R$ whenever $n \geq N(z)$. This follows from the choice of G and from Lemma 3.1.1 of [1]. Since C is compact and since $T_\phi f_n - \tilde{H} - f$ is continuous on C , there is a positive integer N so that this inequality holds for all $z \in C$ and all $n \geq N$. Since $T_\phi f_n - \tilde{H} - f$ is analytic off a compact subset of rD ,

$$(T_\phi f_n)'(\infty) - \tilde{H}'(\infty) - f'(\infty) = (2\pi i)^{-1} \int_C [T_\phi f_n(z) - \tilde{H}(z) - f(z)] \, dz.$$

If n is large, the absolute value of this quantity is at most ϵ , and then (4) implies that $|f'(\infty)| \leq 240r/(r-1) \cdot \gamma(rD - (W \cup E)) + \epsilon$. Letting ϵ go to zero, and then letting f vary, we find that $\gamma(D - W) \leq 240r/(r-1) \cdot \gamma(rD - (W \cup E))$, as desired. This completes the proof that $e \Rightarrow d$ whenever $m(E) = 0$, and completes the proof of Theorem 1.

COROLLARY 1. *Let $W \subset \mathbb{C}^*$ be an open set containing the point ∞ , and let $E \subset \mathbb{C} - W$ be compact. Suppose that for each $z \in E$ we have*

$$\liminf_{\delta \rightarrow 0} \frac{\gamma(D(z, \delta) - (W \cup E))}{\delta} > 0.$$

Then given $f \in H^\infty(W)$ there is a sequence $\{F_n\}_{n=1}^\infty$ in $H^\infty(W)$ such that

- (i) F_n extends analytically to a neighborhood, depending on n , of E ;
- (ii) $|F_n| \leq \|f\|$ on W and on this neighborhood;
- (iii) $\{F_n\}$ converges to f uniformly on each subset of W at a positive distance from E .

PROOF. Using Theorem VIII.5.1 of [4], we see that $R(E) = C(E)$. Hypoth-

esis c of Theorem 1 is satisfied with $r = 1$, and so Corollary 1 follows from the implication $c \Rightarrow e$ of Theorem 1.

It will usually be in the form of this corollary that we make use of Theorem 1. We will give an example later of an instance in which Corollary 1 fails to give any information, while Theorem 1 does tell us what we want to know.

Vitushkin also has methods which allow one to approximate a single function rather than a whole Banach algebra of functions. Most of these methods make the transition from uniform approximation to pointwise bounded approximation. Our main difficulty in using these results is that we are no longer able to reduce norms. This means, for one thing, that we can no longer expect a \liminf condition, as the proof of $c \Rightarrow d$ in Theorem 1 made repeated use of the implication $d \Rightarrow a$ and the fact that the constant in part a is one. Theorem 2 is the analogue of Theorem IV.2.1 and Lemma IV.2.1 of [19], and Theorem 3 is the analogue of Theorem IV.2.2.

THEOREM 2. *Let $W \subset \mathbb{C}^*$ be an open set containing the point ∞ and let $f \in H^\infty(W)$. Let E be a compact set of measure zero which satisfies $\partial W \subset E \subset (C - \bar{W})^- \subset C - W$. (This requires that ∂W have measure zero, and also says that every neighborhood of every point of E meets $C - \bar{W}$.) Then we have the implications $a \Rightarrow b \Leftrightarrow c$, where:*

(a) $\gamma(D - W) \leq m_f \gamma(D - (W \cup E))$ for every open disc D , with the constant m_f independent of the choice of D .

(b) There exist constants $m \geq 1$, $r \geq 1$, $\delta_0 > 0$ with the following property. For every open disc D of radius $\delta < \delta_0$ and for every continuously differentiable function ϕ with compact support contained in D , we have

$$\left| \iint f(z) \operatorname{grad} \phi \, dm(z) \right| \leq m \|f\| \cdot \delta \|\operatorname{grad} \phi\| \cdot \gamma(rD - (W \cup E)).$$

(c) There exists a sequence $\{F_n\}_{n=1}^\infty$ in $H^\infty(W)$ such that

(i) F_n extends analytically to a neighborhood, depending on n , of $W \cup E$;

(ii) $|F_n| \leq M \|f\|$ there, with the constant M independent of n ;

(iii) $\{F_n\}$ converges to f uniformly on each subset of W at a positive distance from E .

PROOF. We will prove that $b \Rightarrow c \Rightarrow b$ and that $a \Rightarrow b$. To show that $b \Rightarrow c$ we make the same changes in Vitushkin's approximation procedure as we did in showing that $d \Rightarrow e$ in Theorem 1. There is only one difference that enters here, and this occurs when we split up the indices and look at those k for which $\gamma(rD_k - (W \cup E)) = 0$. In the present situation, $rD_k - (W \cup E) = rD_k - (\bar{W} \cup E)$ is an open set of zero analytic capacity. It must therefore be empty, and $rD_k \subset W \cup E$. We claim that, in fact, $rD_k \subset W$. For suppose there is a point $z \in rD_k \cap E$. Since rD_k is a neighborhood of $z \in E$, rD_k meets the open set $C - \bar{W}$.

Then there is a disc \tilde{D} contained in the intersection of these two open sets, and

$$\begin{aligned}\tilde{D} - E &= \tilde{D} \cap (C - E) \subset rD_k \cap (C - \bar{W}) \cap (C - E) \\ &= rD_k \cap [C - (\bar{W} \cup E)] = rD_k \cap [C - (W \cup E)] \\ &= rD_k - (W \cup E) = \text{the empty set.}\end{aligned}$$

This contradicts the hypothesis that E has measure zero. Therefore rD_k does not meet E , $rD_k \subset W$, and $D_k \subset W$. Since $D_k - W$ is empty, and f_k is bounded and analytic off a compact subset of $D_k - W$, we must have $f_k \equiv f_k(\infty) = 0$. So we take $g_k \equiv 0$.

Next we show that $c \Rightarrow b$. Let $r = 1$, $\delta_0 = 1$, $m = 8\pi M$, where M is the constant of part c. Choose D and ϕ . If $\|\text{grad } \phi\|$ is 0 or ∞ there is nothing to prove, so we assume it is equal to q , with $0 < q < \infty$. Let $\epsilon > 0$. Choose $\eta > 0$ so that N , the η -neighborhood of E , has measure $m(N) \leq \epsilon/2(M+1)q\|f\|$. Next choose F to be one of the F_n so that F is analytic on $G \supset W \cup E$, $|F| \leq M\|f\|$ on G , and $|F - f| \leq \epsilon/2\pi q\delta^2$ on $W - N$. Define $H \equiv F$ on $W \cup (G \cap N)$ and $H \equiv 0$ elsewhere. Also define $f \equiv 0$ off W . Consider the function $T_\phi H$. This is analytic off a compact subset of $D - (W \cup E)$ and is bounded by $8\delta q M\|f\|$. We have

$$\begin{aligned}(T_\phi H)'(\infty) &= \frac{1}{2\pi i} \int_{\partial D} \frac{1}{\pi} \iint \frac{H(w)}{w-z} \text{grad } \phi \, dm(w) \, dz \\ &= \frac{1}{\pi} \iint_D H(w) \text{grad } \phi \, dm(w).\end{aligned}$$

Now,

$$\begin{aligned}\iint_D [f(w) - H(w)] \text{grad } \phi \, dm(w) &= \iint_{D \cap [W \cup (G \cap N)]} \\ &= \iint_{(D \cap [W \cup (G \cap N)]) - N} + \iint_{D \cap [W \cup (G \cap N)] \cap N}\end{aligned}$$

From the three inclusions:

$$(D \cap [W \cup (G \cap N)]) - N \subset W - N,$$

$$D \cap [W \cup (G \cap N)] \subset W \cup (G \cap N) = [W \cup (G \cap N)] \cap G,$$

$$D \cap [W \cup (G \cap N)] \cap N \subset N,$$

we see that the last two integrals above are bounded in absolute value by $\epsilon/2\pi q\delta^2 \cdot q\pi\delta^2 = \epsilon/2$ and $(M+1)\|f\| \cdot q \cdot \epsilon/2(M+1)q\|f\| = \epsilon/2$ respectively. Therefore

$$\begin{aligned}
\left| \iint f(w) \operatorname{grad} \phi \, dm(w) \right| &= \left| \iint_D f(w) \operatorname{grad} \phi \, dm(w) \right| \\
&\leq \left| \iint_D H(w) \operatorname{grad} \phi \, dm(w) \right| + \left| \iint_D [f(w) - H(w)] \operatorname{grad} \phi \, dm(w) \right| \\
&\leq \pi |(T_\phi H)'(\infty)| + \epsilon \leq 8\pi\delta qM\|f\|\gamma(D - (W \cup E)) + \epsilon \\
&= m\|f\| \cdot \delta \|\operatorname{grad} \phi\| \cdot \gamma(D - (W \cup E)) + \epsilon.
\end{aligned}$$

Since ϵ is arbitrary we are finished. Thus b and c are equivalent.

Now suppose that a is true. Let $r = 1$, $\delta_0 = 1$, $m = 8m_f$. Choose D and ϕ . Extend f to \mathbb{C}^* by setting it equal to zero off W . Then $T_\phi f$ is defined, is analytic off a compact subset of $D - W$, and is bounded by $8\delta \|\operatorname{grad} \phi\| \|f\|$. Thus $|(T_\phi f)'(\infty)| \leq 8\delta \|\operatorname{grad} \phi\| \|f\| \gamma(D - W) \leq 8m_f \|f\| \cdot \delta \|\operatorname{grad} \phi\| \cdot \gamma(D - (W \cup E))$. Since $(T_\phi f)'(\infty)$ is the integral in b, b is true. This completes the proof of Theorem 2.

We remark that it is possible to replace the hypothesis that E have measure zero by the less restrictive requirement that E have no interior. This allows us to include situations in which ∂W may have positive area. In this case the conclusion of Theorem 2 is that $a \Rightarrow b \Rightarrow c$, and the proof proceeds as above.

In Vitushkin's work the extra condition on E does not explicitly appear, but it is there. If K is a compact set, Vitushkin considers $E = \partial K$, and the condition reduces to the tautology $\partial(\text{interior of } K) \subset \partial K \subset (\mathbb{C} - (\text{interior of } K)^-)^-$.

Finally we note that we cannot show equivalence between part a and either b or c; knowing about the one function f is just not enough.

THEOREM 3. *Let f be continuous on \mathbb{C}^* with modulus of continuity $\omega(\delta)$, and let $E \subset \mathbb{C}^*$ be any set. Assume that there exist numbers $\tilde{m} \geq 1$, $\tilde{\delta}_0 > 0$ with the property that $|\int_{\partial Q} f(z) dz| \leq \tilde{m}\omega(\delta)\gamma(Q - E)$ for all squares Q of side $\delta < \tilde{\delta}_0$. Then there exist constants $m \geq 1$, $r \geq 1$, $\delta_0 > 0$ with the following property. For every open disc D of radius $\delta < \delta_0$ and every continuously differentiable function ϕ with compact support contained in D , we have*

$$\begin{aligned}
\left| \iint f(z) \operatorname{grad} \phi \, dm(z) \right| &\leq m\omega(2\delta) \cdot \delta \|\operatorname{grad} \phi\| \cdot \gamma(rD - E) \\
&\leq 2m\|f\| \cdot \delta \|\operatorname{grad} \phi\| \cdot \gamma(rD - E).
\end{aligned}$$

This is proved exactly as Theorem IV.2.2 of [19], and continuity is essential. Vitushkin, who works with uniform approximation, is able to obtain a converse. From a statement about pointwise bounded approximation, which is what we have here according to Theorem 2, we have not been able to produce a converse.

2. When does $\gamma(V) = \gamma(\bar{V})$? There are examples of bounded open sets V with $\gamma(V) < \gamma(\bar{V})$. However we can use Corollary 1 to obtain equality in many cases.

THEOREM 4. *Let V be a bounded open set. If at least one of the following conditions holds, then $\gamma(V) = \gamma(\bar{V}) = \alpha(V) = \alpha(\bar{V})$.*

- (a) *V has finitely many components.*
- (b) *The diameters of the components of V are bounded away from zero.*
- (c) *∂V coincides with the union of the boundaries of the components of V .*
- (d) *Every point of ∂V is either a boundary point of a component of V , or is a cluster point of a sequence of components of V whose diameters are bounded away from zero.*

Moreover, conditions c and d can be weakened to allow part of ∂V to be what Gamelin and Garnett [6, §7] call a σ -curvilinear null set, that is, a countable union of bounded sets B_j , each of which lies on a C^2 curve C_j , and has zero outer arc length. (Any countable set is a σ -curvilinear null set, for example.)

PROOF. Let $W = \mathbb{C}^* - \bar{V}$ and let $E = \partial V$. Let $z \in E$. If z is a boundary point of one or more components of V , choose one such component and let δ_0 be the diameter of that component. If z does not belong to the boundary of any component of V , choose a sequence of components which cluster at z , and whose diameters are bounded away from zero, and let δ_0 be the infimum of those diameters. If $0 < \delta < \delta_0$ then $D(z, \delta) \cap V$ contains an arc A of diameter $\frac{1}{2}\delta$. Since the analytic capacity of a continuum is at least one fourth the diameter, $\gamma(D(z, \delta) - (W \cup E))/\delta = \gamma(D(z, \delta) \cap V)/\delta \geq \gamma(A)/\delta \geq 1/8$. Thus Corollary 1 applies.

Let $V \subset D(0, R)$ and let C be the circle with center 0 and radius $2R$. Let f be the Ahlfors function for \bar{V} and let $\epsilon > 0$. Let F be one of the functions F_n of Corollary 1 so that $\|F - f\|_C \leq \epsilon/2R$. Then F is "admissible" for a compact subset of V , and $\gamma(\bar{V}) = f'(\infty) \leq |F'(\infty)| + \epsilon \leq \gamma(V) + \epsilon$. Since ϵ is arbitrary, $\gamma(\bar{V}) \leq \gamma(V)$, and therefore $\gamma(\bar{V}) \leq \gamma(V) = \alpha(V) \leq \alpha(\bar{V}) \leq \gamma(\bar{V})$.

For the case of finitely many components we mention two other proofs. The first appears in [15] and uses completely different methods. The second was communicated to the author by T. Gamelin. The condition of Theorem 7(vi) of [5], with $K = \mathbb{C}^* - V$, E = the empty set, is satisfied and so $R(K)$ is a hypo-dirichlet algebra on $\partial K = \partial V$. From here one can proceed as before, approximating the Ahlfors function for \bar{V} , using part (iv) of that theorem. Gamelin has also pointed out that in cases a and b it is possible to use Corollary VIII.10.6 of [4] to produce a sequence of approximating functions. Finally, all four conditions can be taken care of by Theorem 8.9 and the last paragraph on p. 389 of [6], and here is where the remark about σ -curvilinear null sets enters.

Since we plan to concentrate next on the types of boundary points described in this theorem, we define, for any bounded open set V , $S = S(V) =$ the union of the boundaries of the components of V , and $S_0 = S_0(V) =$ the union of $S(V)$ and all boundary points of V which are cluster points of sequences of components of V whose diameters are bounded away from zero. For convenience we also introduce the notation $N(A, \delta)$ for the δ -neighborhood of the set $A \subset \mathbb{C}$.

We have just seen that the situation in which the boundary of V was equal to S or S_0 was a very nice one. We were able to add ∂V on to V , or, equivalently, remove ∂V from \bar{V} , without affecting the numerical value of the analytic capacity. Of course it follows that *part* of ∂V can also be added on to V or subtracted from \bar{V} with the same result. Now we consider the situation in which ∂V is *not* equal to S_0 . Again, there are examples of open sets V with $\gamma(V) < \gamma(\bar{V})$, so it is too much to hope for the possibility that *all* of ∂V can be added or removed. But what about *part* of ∂V ? Specifically, what about the set S_0 ? On S_0 the \liminf condition is satisfied, so we would expect that S_0 can be added or removed. However S_0 is not a compact set because $S_0 \neq \partial V = \bar{S}_0$, and so the results above do not apply to S_0 . They do apply to compact subsets K of S_0 , and so we have the result that $\gamma(\bar{V}) = \gamma(\bar{V} - K)$.

Is it also true that $\gamma(V) = \gamma(V \cup K)$? This is not immediately obvious. The trouble here is that $V \cup K$ is not compact, and so to look at $\gamma(V \cup K)$ we must look at $\gamma(L)$ where L is a compact subset of $V \cup K$. The set L may not contain enough of V to produce the arcs which previously gave us our capacity estimate.

What we plan to do in this section is to show that $\gamma(V) = \gamma(V \cup S_0)$ and $\gamma(\bar{V}) = \gamma(\bar{V} - S_0)$ for all bounded open sets V . This will include the above statements about $K \subset S_0$. The following lemma and Theorem 5 were given by the referee, Donald Wilken, and improve the author's original Theorem 5, which had been stated and proved only for a certain class of open sets. These are the sets of type I, which will be defined later.

LEMMA. *Let V be a bounded open set and let $L \subset V \cup S_0(V)$ be compact. Given $\delta > 0$, there exists a bounded open set U with*

- (a) $U \subset V \cap N(L, \delta)$;
- (b) $L \subset \bar{U}$, which says that U is not empty if L is not empty;
- (c) $\partial U = S_0(U)$, which says $\gamma(U) = \gamma(\bar{U}) = \alpha(U) = \alpha(\bar{U})$ via Theorem 4d.

PROOF. Let O_n be the union of all components of V with diameters at least δ/n . Then $V \cup S_0(V) = \bigcup \bar{O}_n$. Let $L_n = L \cap \bar{O}_n$ and $G_n = V \cap N(L_n, \delta/n)$. Let $\{U_{n,j}\}$ be the collection of all components of U_n with diameters at least $\delta/2n$, and let U_n be the union of this collection. Let U be the

union of the U_n . Part a follows immediately; part b follows from the relation $L_n \subset \bar{U}_n$; and part c follows from the relation $\partial U_n = S_0(U_n)$.

THEOREM 5. *If V is any bounded open set, then $\gamma(V) = \gamma(V \cup S_0) = \alpha(V) = \alpha(V \cup S_0)$ and $\gamma(\bar{V}) = \gamma(\bar{V} - S_0)$.*

PROOF. The first string of equalities follows immediately from the lemma. Given $L \subset V \cup S_0$ compact, we take $\delta = 1$ and the lemma yields a set U with $\gamma(L) \leq \gamma(\bar{U}) = \gamma(U) \leq \gamma(V)$. Taking the supremum over L we see that $\gamma(V \cup S_0) \leq \gamma(V) = \alpha(V) \leq \alpha(V \cup S_0) \leq \gamma(V \cup S_0)$. The second part of the theorem requires some work.

Let $\epsilon > 0$, let $V \subset D(0, R)$, and let C be the circle with center at the origin and radius $2R$. For each positive integer n , let O_n be the union of all those components of V with diameters at least $1/n$. Then $O_1 \subset O_2 \subset O_3 \subset \dots$.

We apply Corollary 1 with $W_1 = C^* - \bar{V}$, $E_1 = \partial O_1$, and $f = g_1 = \text{Ahlfors function for } \bar{V}$. This yields a function g_2 , which is one of the $\{F_n\}$, such that g_2 is analytic in a neighborhood $G_2 \supset W_1 \cup \partial O_1$, with $|g_2| \leq 1$ on G_2 and $\|g_2 - g_1\|_C \leq \epsilon/4R \cdot 1/2$. We apply the lemma with V , $L_2 = \partial O_2 - G_2$, and $\delta_2 = 1/4 \text{ dist}(\partial O_1, \partial G_2)$. This yields a set $U_2 \subset V \cap N(L_2, \delta_2)$ with $L_2 \subset \bar{U}_2$ and $\partial U_2 = S_0(U_2)$.

We set $W_2 = G_2 - \bar{U}_2$, $E_2 = \partial U_2 - W_2$ and we observe that:

- (i) $W_2 \cup E_2 = (G_2 - \bar{U}_2) \cup \partial U_2 = (G_2 - U_2) \cup \partial U_2 \subset C^* - U_2$;
- (ii) $W_1 = W_1 - \bar{V} \subset G_2 - \bar{V} \subset G_2 - \bar{U}_2 = W_2$;
- (iii) $\partial O_1 \cup \partial O_2 = \partial O_2 \subset W_2 \cup E_2$.

Corollary 1 applies to W_2, E_2, g_2 because every $z \in E_2 \subset S_0(U_2)$ satisfies $\gamma(D(z, \delta) - (W_2 \cup E_2)) \geq \gamma(D(z, \delta) \cap U_2) \geq \delta/8$ for small positive $\delta < \delta(z)$.

When we apply Corollary 1, we obtain a function g_3 analytic on an open set $G_3 \supset W_2 \cup E_2 \supset W_1 \cup \partial O_1 \cup \partial O_2$ with $|g_3| \leq 1$ on G_3 and $\|g_3 - g_2\|_C \leq \epsilon/4R \cdot 1/4$. Next we apply the lemma with V , $L_3 = \partial O_3 - G_3$, and $\delta_3 = \min\{1/2\delta_2, 1/4 \text{ dist}(\partial O_1 \cup \partial O_2, \partial G_3)\}$, and this yields a set $U_3 \subset V \cap N(L_3, \delta_3)$ with $L_3 \subset \bar{U}_3$ and $\partial U_3 = S_0(U_3)$.

We then set $W_3 = G_3 - \bar{U}_3$ and $E_3 = \partial U_3 - W_3$ and proceed as above. We find (i) $W_3 \cup E_3 \subset C^* - U_3$; (ii) $W_1 \subset W_3$; (iii) $\partial O_1 \cup \partial O_2 \cup \partial O_3 = \partial O_3 \subset W_3 \cup E_3$.

Continuing in this manner, we define $d_n = 1/2 \text{ dist}(\partial O_n, \partial G_{n+1}) = 1/2 \text{ dist}(\partial O_1 \cup \dots \cup \partial O_n, \partial G_{n+1})$ and

$$\delta_n = \min\{1/2\delta_{n-1}, 1/4 \text{ dist}(\partial O_1 \cup \dots \cup \partial O_{n-1}, \partial G_n)\},$$

and this produces a sequence $\{g_n\}_{n=1}^\infty$ of functions and a sequence $\{W_n\}_{n=1}^\infty$ of open sets which satisfy the following conditions:

- (a) $(C^* - \bar{V}) \cup \partial O_1 \cup \dots \cup \partial O_n = W_1 \cup \partial O_n \subset W_{n+1}$;

(b) g_n is analytic and bounded by 1 in W_n ;

(c) $\|g_n - g_{n-1}\|_C \leq \epsilon/4R \cdot 2^{-n}$;

(d) $N(\partial O_n, d_n)$ is contained in each of the sets W_{n+1}, W_{n+2}, \dots .

Only the last of these assertions requires any elaboration, and it is verified without much difficulty by the choice of d_n and δ_n , and by the construction of the sets W_n as:

$$W_1 \cup \partial O_1 \cup \partial O_2 \cup \dots \cup \partial O_n \subset W_n \cup E_n \subset G_{n+1};$$

$$W_{n+1} = G_{n+1} - \bar{U}_{n+1};$$

$$W_1 \cup \partial O_1 \cup \partial O_2 \cup \dots \cup \partial O_{n+1} \subset W_{n+1} \cup E_{n+1} \subset G_{n+2};$$

$$W_{n+2} = G_{n+2} - \bar{U}_{n+2}.$$

Now we set $W = (C^* - \bar{V}) \cup \bigcup N(\partial O_n, d_n)$ and we take g to be any cluster point of the sequence $\{g_n\}$, a normal family of functions. In each set $N(\partial O_n, d_n)$ the functions $g_{n+1}, g_{n+2}, g_{n+3}, \dots$ are all analytic, and therefore g is analytic in W . Also, g is bounded by 1 on W , so g is "admissible" for the compact set $C^* - W$, when we extend g to C^* by setting it equal to zero off W . Note that $C^* - \bar{V} \subset W$ implies $C^* - W \subset \bar{V}$, and since the set $S_0 = \bigcup \partial O_n$ is contained in W , we have $C^* - W \subset \bar{V} - S_0$. Thus g is "admissible" for a compact subset of $\bar{V} - S_0$.

It is easy to see that $\|g - f\|_C \leq \epsilon/2R$, from which it follows that $\gamma(\bar{V}) = f'(\infty) \leq |g'(\infty)| + \epsilon \leq \gamma(\bar{V} - S_0) + \epsilon$. Since ϵ is arbitrary, the proof of Theorem 5 is complete.

One example worth looking at in the light of Theorem 5 is Example 3.11 of [20]. Here L. Zalcman describes a bounded open set U with $\gamma(U) \neq \gamma(\bar{U})$. The set U is a union of open discs with pairwise disjoint closures, and these discs cluster only on the line segment $L = [0, 1]$. In this example, the diameters of the components of U , i.e., the diameters of the discs, converge to zero. Therefore $S_0 = S$ and Theorem 5 tells us that $\gamma(\bar{U}) = \gamma(\bar{U} - S) = \gamma(U \cup L)$. Thus the only reason that $\gamma(U) \neq \gamma(\bar{U})$ is that $\gamma(U) \neq \gamma(U \cup L)$, which is what we expected.

We wish now to describe some special classes of open sets.

DEFINITION. Let V be a bounded open set with components $\{V_j\}_{j=1}^\infty$. Suppose that the components of V can be grouped together as

$$V_1, V_2, \dots, V_{j_1}; V_{j_1+1}, V_{j_1+2}, \dots, V_{j_2}; \dots;$$

$$V_{j_n+1}, V_{j_n+2}, \dots, V_{j_{n+1}}; \dots$$

and that this grouping has the property that, for each n , the sets $\bigcup_{j=j_n+1}^{j_{n+1}} V_j$ and $\bigcup_{j=j_{n+1}+1}^\infty V_j$ have disjoint closures. Then we will call V a bounded open set of type I.

For a set V of type I, the "disjoint closures" condition tells us that the

set $\partial V - S$ is compact. That is, S is relatively open in ∂V , hence $\partial V - S$ is relatively closed in ∂V , which says that $\partial V - S$ is a compact set in the plane. For arbitrary open sets V , $\partial V - S$ need not be closed. Probably the simplest example is the following. Let $\{D_j\}_{j=1}^{\infty}$ be a collection of open discs with pairwise disjoint closures, such that (i) $\bar{D}_j \subset D(0, 1)$, (ii) \bar{D}_j is contained in the upper half plane $\{z \in \mathbb{C} : \text{Im } z > 0\}$, and (iii) $\bigcup_{j=1}^{\infty} D_j$ clusters only on the set $\{0\} \cup \{1/2, 1/3, 1/4, \dots\}$. Let $D_0 = D(-i, 1)$ and let $V = \bigcup_{j=0}^{\infty} D_j$. Then $\partial V - S = \{1/2, 1/3, 1/4, \dots\}$.

DEFINITION. Let V be bounded open set of type I. If $\partial V - S$ is a (compact) set of zero analytic capacity, then we call V a bounded open set of type II. If, in addition, each component of V is a square, then we call V a bounded open set of type III. Now let V be *any* bounded open set. If $(\partial V - S_0)$ is a set of zero analytic capacity, then we call V a bounded open set of type S_0 .

We have the inclusion relations $\text{III} \subset \text{II} \subset \text{I} \cap S_0$. Each of these inclusions is proper, and there is no inclusion relation between I and S_0 . The example just described is a set of type S_0 which is not of type I, and we will later describe a set of type I which is not of type S_0 . A sequence of rectangles of width 1, each contained in $D(0, 2)$, clustering only on the unit interval, affords an example of a set which is of types I and S_0 but not of type II.

THEOREM 6. *The following conjectures are equivalent.*

- (a) *The two set functions α and γ are semi-additive (for disjoint compact sets).*
- (b) $\gamma(V) = \gamma(\bar{V})$ *for all bounded open sets of type S_0 .*
- (c) $\gamma(V) = \gamma(\bar{V})$ *for all bounded open sets of type II.*
- (d) $\gamma(V) = \gamma(\bar{V})$ *for all bounded open sets of type III.*

PROOF. We first show $d \Rightarrow a$. Assuming that γ is not semi-additive, A. Davie [3, Theorem 5.2] constructs a compact set E of zero analytic capacity, and a bounded set F , which is a countable union of closed, nonoverlapping squares, such that $E \cup F = \bar{F}$ and $\gamma(F) < \gamma(E \cup F)$. Let V be the corresponding union of open squares. In the construction, Davie has disjoint compact sets E_n , F_n at each stage, and each of these is a finite union of nonoverlapping closed squares. He then places $F_{n+1} - F_n$ in the interior of E_n (and also places E_{n+1} in the interior of E_n) and he later defines $F = \bigcup_{n=1}^{\infty} F_n$. We may then take the groupings in the definition of sets of type I to be: squares of F_1 ; squares of $(F_2 - F_1)$; squares of $(F_3 - F_2)$; \dots . Thus V is a set of type I, and since the set $\partial V - S$ is equal to E , and each component of V is a square, V is also a set of type III. Finally $\gamma(V) = \gamma(V \cup S) = \gamma(F) < \gamma(E \cup F) = \gamma(\bar{V})$. This shows that d implies the semi-additivity of γ , and then d implies a, because the semi-additivity of γ is equivalent to that of α .

Now assume that γ is semi-additive. Let V be a set of type S_0 and let $A = \partial V - S_0$, a set of zero analytic capacity. We must have $\gamma(V) = \gamma(E \cup V)$ for all compact sets E of zero analytic capacity, hence for all compact sets $E \subset A$. We claim that this implies $\gamma(V) = \gamma(A \cup V)$, from which it will follow that $\gamma(V) = \gamma(A \cup V) = \gamma(\bar{V} - S_0) = \gamma(\bar{V})$ by Theorem 5.

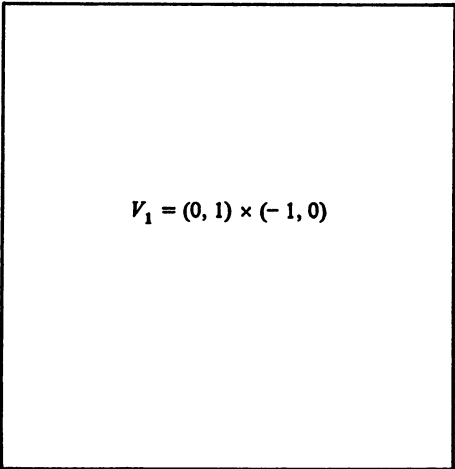
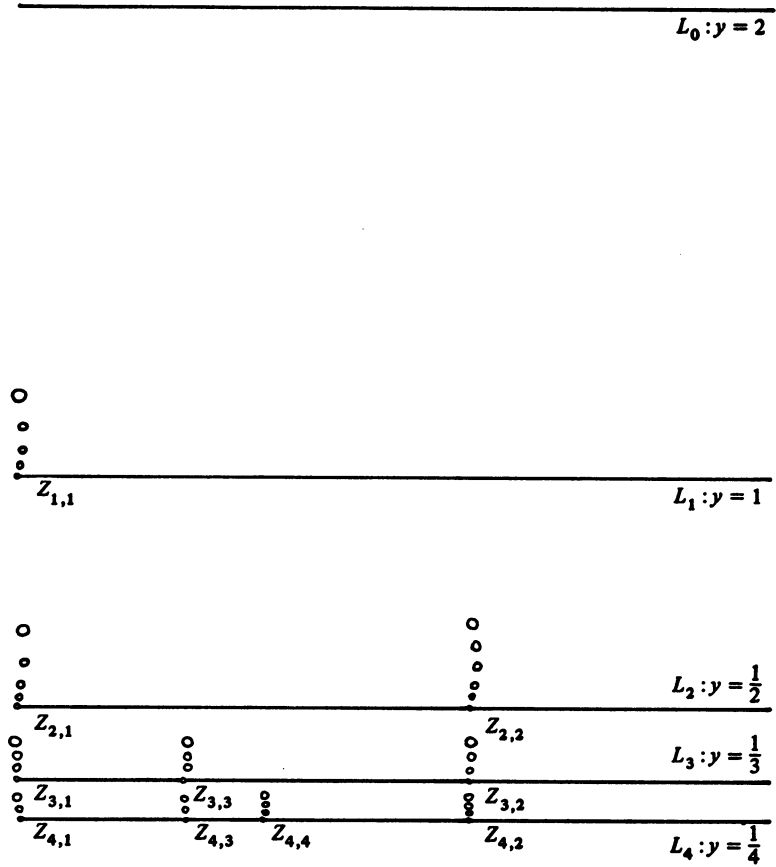
Let $L \subset A \cup V$ be compact. Then $L \subset V \cup (L \cap \partial V)$. Since $L \cap \partial V$ is a compact subset of A , we have $\gamma(L) \leq \gamma(V \cup (L \cap \partial V)) = \gamma(V)$. Taking the supremum over L we see that $\gamma(A \cup V) \leq \gamma(V) \leq \gamma(A \cup V)$. This proves $a \Rightarrow b$, and the final implications, $b \Rightarrow c \Rightarrow d$, are trivial.

3. Examples. For a set V of type II, $\partial V - S$ is a compact set of zero analytic capacity. For a set V of type S_0 , $\partial V - S_0$ has zero analytic capacity, but does that mean that $(\partial V - S_0)^-$ is a compact set of zero capacity? Our first example shows that this need not be true. Let V_1 be the square $(0, 1) \times (-1, 0)$. Let $\{r_m\}_{m=1}^\infty$ be an enumeration of the rationals in $[0, 1]$. Let L_0 be the line $y = 2$ in the xy -plane, and for each positive integer n , let L_n be the line $y = 1/n$. For each pair of positive integers m and n , with $m \leq n$, let $z_{n,m}$ be the point $(r_m, 1/n)$. See the figure. Let $\{D_{n,m,j}\}_{j=1}^\infty$ be a sequence of discs with pairwise disjoint closures contained in the open strip between L_{n-1} and L_n such that (i) these discs cluster only at the point $z_{n,m}$ and (ii) $D_{n,m,j}$ has radius less than $2^{-n-m-j-3}$. We can do this for each pair of positive integers m and n with $m \leq n$, and we can insure that all discs $D_{n,m,j}$ have pairwise disjoint closures. Let V be the set $V = \bigcup_{n=1}^\infty \bigcup_{m=1}^n \bigcup_{j=1}^\infty D_{n,m,j} \cup V_1$.

Then we have $S = S_0$, and $\partial V - S = \{z_{n,m} : m = 1, \dots, n; n = 1, 2, \dots\}$. Since $\partial V - S$ is countable it has zero analytic capacity. However its closure contains the interval $[0, 1]$ and has capacity $1/4$. In this particular example we do have $\gamma(V) = \gamma(\bar{V})$ via Theorem 4c, because only countably many boundary points of V fail to belong to the set S .

If we replace V by $U = V - V_1$ then the discs of U still cluster on $[0, 1]$. Estimating $\gamma(U)$ by perimeter/ 2π according to Proposition 3.8 of [20], we see that $\gamma(U) \leq 1/8 < 1/4 = \gamma([0, 1]) \leq \gamma(\bar{U})$. The set U is no longer of type S_0 and so the fact that $\gamma(U) \neq \gamma(\bar{U})$ is not disturbing.

Now we would like to look at some examples of sets V of types I and II. Sets of type I include Zalcman's Example 3.11 and the set U just described, so we cannot hope to have $\gamma(V) = \gamma(\bar{V})$ for all such sets. The first set we describe will be a set of type I which is not of type II or of type S_0 . All five parts of Theorem 1 will fail for this set, and we will not have $\gamma(V) = \gamma(\bar{V})$. The major example of this section will be a set of type II for which all five parts of Theorem 1 are true, and therefore $\gamma(V) = \gamma(\bar{V})$. However if we were to attempt to show $\gamma(V) = \gamma(\bar{V})$ for this set by using Corollary 1 instead of Theorem 1, we would not be able to do so. The lim inf condition of Corollary 1 is not precise



enough. We will also describe a set of type II for which we do not know whether $\gamma(V) = \gamma(\overline{V})$. We will be able to relate $\alpha(V)$ and $\alpha(\overline{V})$, though. Each of the open

sets to be described will have $S = S_0$ because the diameters of the components will converge to zero.

Let $E = \bigcap_{n=0}^{\infty} E_n$, where E_0 is the closed unit square, E_n consists of 4^n closed squares of side 3^{-n} , and each component of E_n contains four components of E_{n+1} , these being the four corner squares of side 3^{-n-1} . The components of E_n will be indexed as $E_{n,j}$ ($1 \leq j \leq 4^n$) and we take $z_{n,j}$ to be the center of $E_{n,j}$. Let $0 < r_n \leq 1/6 \cdot 3^n$ and let $D_{n,j} = D(z_{n,j}, r_n)$. If we take V to be the union of all $D_{n,j}$ then V is a bounded open set of type I. The set E is the product of the usual Cantor set with itself, and according to [18], $\gamma(E) \geq \alpha(E) > 0$. So V is not a set of type II or of type S_0 . If the r_n are chosen very small, we find, as in Zalcman's example, that $\gamma(V) \leq \Sigma 4^n r_n < \gamma(E) \leq \gamma(\bar{V})$. This tells us that part e of Theorem 1 fails for $W = C^* - \bar{V}$ and our set E . So all parts of that theorem fail, and there would be no hope in looking at the ratio of capacities. On the other hand, if we take $r_n = 1/6 \cdot 3^n$, then the discs would shrink at the same rate as the squares $E_{n,j}$, and we would get a capacity estimate giving $\gamma(V) = \gamma(\bar{V})$. (For $z \in E$ and for any positive δ , we can always construct an arc in $D(z, \delta) \cap V$ whose diameter, divided by δ , stays away from zero.)

Now let us keep the same notation, but use it in a different situation. We take the squares of E_n to have side 5^{-n} and we take $0 < r_n \leq 5^{-n-1}$. We again get a set V of type I, but this time V is also of type II. To see that, we estimate $\gamma(E_n)$ by means of Proposition 3.8 of [20]. We have perimeter of $E_n = 4^n \cdot 4 \cdot 5^{-n}$, so perimeter of $E_n/2\pi \leq (4/5)^n$. Since $E \subset E_n$ for all n , $\gamma(E) \leq \text{perimeter of } E/2\pi = 0$. Thus V is of type II, and we hope that $\gamma(V) = \gamma(\bar{V})$. There are two ways in which we can show that this is so. The first method modifies the proof of Theorem 3.1 of [7] to treat pointwise bounded approximation instead of uniform approximation. This is the sort of thing we have been doing in §1 and it does apply to Garnett's proof, since our set E is his set $E_{0.6}$. The second method is the estimation of capacities. This depends on knowing the r_n . We will give an example of these estimates, because it will show a situation in which Corollary 1 would fail to provide any information, while Theorem 1 is successful.

We take $r_n = 25^{-n-1}$, and we claim that for each $z \in E$,

$$\liminf_{\delta \rightarrow 0} \frac{\gamma(D(z, \delta) \cap V)}{\gamma(D(z, \delta) \cap \bar{V})} > 0$$

but

$$\lim_{\delta \rightarrow 0} \frac{\gamma(D(z, \delta) \cap (V \cup S))}{\delta} = 0.$$

Choose δ with $0 < \delta < 1/10$ and fix $z \in E$. Then there is a positive integer N such that $D(z, \delta)$ contains a square of E_N but does not contain any square of E_{N-1} . This tells us that N and δ are related by $\sqrt{2}/5^N < 2\delta$ and

$\delta < \sqrt{2}/5^{N-1} = 5\sqrt{2}/5^N$. That is,

$$(1) \quad \sqrt{2}/2 \cdot 5^{-N} < \delta < 5\sqrt{2} \cdot 5^{-N}.$$

Let Q be the square of E_N which contains z . Since $D(z, \delta)$ contains some square (or squares) of E_N , it follows that $Q \subset D(z, \delta)$. Next let Q' be the square of E_{N-1} which contains Q . The distance from $z \in Q'$ to any *other* square of E_{N-1} is at least $3/5^{N-1} > \delta$. Therefore $D(z, \delta)$ meets no square of E_{N-1} other than Q' , and

$$(2) \quad D(z, \delta) \cap (V \cup S) \cap E_{N-1} = D(z, \delta) \cap (V \cup S) \cap Q',$$

$$(3) \quad D(z, \delta) \cap \bar{V} \cap E_{N-1} = D(z, \delta) \cap \bar{V} \cap Q'.$$

Let $A = \bigcup_{m=0}^{N-2} \bigcup_{k=1}^{4^m} \bar{D}_{m,k}$. Among those discs which make up A , it is possible for only one of them to meet $D(z, \delta)$. Such a disc, if there is one, will be $\bar{D}_{N-2,j}$ for some j , and this will be *the* disc nearest to the square Q' of E_{N-1} . That is, it will be the disc with the same center as Q' , the square of E_{N-2} which contains Q' . (Note that (1) implies $\sqrt{2}/2 \cdot 5^N < \delta < 1/10$, or $5\sqrt{2} < 5^N$, or $2 \leq N$. Thus $N-2$ as a subscript is meaningful.) Therefore

$$(4) \quad D(z, \delta) \cap (V \cup S) \cap A \subset D(z, \delta) \cap (V \cup S) \cap \bar{D}_{N-2,j} \subset \bar{D}_{N-2,j},$$

$$(5) \quad D(z, \delta) \cap \bar{V} \cap A \subset D(z, \delta) \cap \bar{V} \cap \bar{D}_{N-2,j} \subset \bar{D}_{N-2,j}.$$

Since $V \cup S \subset \bar{V} \subset E_{N-1} \cup A$ we find from (2) through (5) that

$$(6) \quad D(z, \delta) \cap (V \cup S) \subset [D(z, \delta) \cap (V \cup S) \cap Q'] \cup \bar{D}_{N-2,j},$$

$$(7) \quad D(z, \delta) \cap \bar{V} \subset [D(z, \delta) \cap \bar{V} \cap Q'] \cup \bar{D}_{N-2,j}.$$

We estimate the capacities of these sets by perimeter/2 π .

From (6) we see that $D(z, \delta) \cap (V \cup S)$ is contained in the union of the following closed discs:

one disc, $\bar{D}_{N-2,j}$, of radius 25^{-N+1} ;

one disc, say $\bar{D}'_{N-1,1}$, of radius 25^{-N} ;

four discs, say $\bar{D}'_{N,1}, \dots, \bar{D}'_{N,4}$, each of radius 25^{-N-1} ;

16 discs, say $\bar{D}'_{N+1,1}, \dots, \bar{D}'_{N+1,16}$, each of radius 25^{-N-2} ; etc.

Note that each disc $\bar{D}'_{N+m,k}$ is one of the $D_{N+m,l}$, for $m = -1, 0, 1, 2, \dots$; $k = 1, 2, \dots, 4^{m+1}$; $l = 1, 2, \dots, 4^{N+m}$. Adding the radii, we get $25^{-N}(25 + 25/21) < 50/25^N$. Thus

$$(8) \quad \gamma(D(z, \delta) \cap (V \cup S)) < 100 (\sqrt{2}/2 \cdot 5^N)^2 < 100 \delta^2.$$

In the same way, $D(z, \delta) \cap \bar{V}$ is contained in the union of: one disc, $\bar{D}_{N-2,j}$; one disc, $\bar{D}'_{N-1,1}$; four discs $\bar{D}'_{N,1}, \dots, \bar{D}'_{N,4}$; etc.; 4^{n-1} discs, $\bar{D}'_{N+n-2,1}, \dots, \bar{D}'_{N+n-2,4^{n-1}}$; and 4^n squares, $E_{N+n-1,k}$ ($1 \leq k \leq 4^n$),

where n is any positive integer whatsoever. Then $\gamma(D(z, \delta) \cap \bar{V}) \leq 50/25^N + 4^n \cdot 4 \cdot 5^{-N-n+1}/2\pi \leq 100\delta^2 + 4 \cdot 5^{-N}(4/5)^n$. We will choose $n \geq 8N$, because $(4/5)^8 < 1/5$, and so $(4/5)^n \leq (4/5)^{8N} \leq (1/5)^N$. Therefore

$$(9) \quad \gamma(D(z, \delta) \cap \bar{V}) \leq 100\delta^2 + 4 \cdot 25^{-N} \leq 108\delta^2.$$

One more estimate we need is a lower bound for $\gamma(D(z, \delta) \cap V)$. Since $D(z, \delta)$ contains a square of E_N , $D(z, \delta) \cap V$ contains the disc concentric with this square. The radius of the disc is $25^{-N-1} = (25 \cdot 50)^{-1}(50 \cdot 25^{-N}) \geq (1250)^{-1}\delta^2$. Therefore

$$(10) \quad \gamma(D(z, \delta) \cap V) \geq \delta^2/1250.$$

From (9) and (10) we have, for $z \in E$ and $0 < \delta < 1/10$, $\gamma(D(z, \delta) \cap V)/\gamma(D(z, \delta) \cap \bar{V}) \geq (1250 \cdot 108)^{-1} \geq 10^{-6}$, so that

$$\liminf_{\delta \rightarrow 0} \frac{\gamma(D(z, \delta) \cap V)}{\gamma(D(z, \delta) \cap \bar{V})} \geq 10^{-6} > 0.$$

However, as δ goes to zero, $0 \leq \liminf \gamma(D(z, \delta) \cap (V \cup S))/\delta \leq \limsup \gamma(D(z, \delta) \cap (V \cup S))/\delta \leq \limsup 100\delta = 0$. Theorem 1, with $W = \mathbb{C}^* - \bar{V}$, and with E (of Theorem 1) $= \partial V = E \cup S$, shows that $\gamma(V) = \gamma(\bar{V})$. The short form, Corollary 1, gives no information at all, even if we take E (in Corollary 1) to be the Cantor set E instead of the larger set, ∂V .

One more Cantor set is the one in which E_n consists of squares of side 4^{-n} . If we take $0 < r_n \leq 4^{-n-1}$, then V is a set of type I. Proving that $\gamma(E) = 0$, and hence that V is a set of type II, is very difficult, but John Garnett has shown in [8] that this is so. So again we hope that $\gamma(V) = \gamma(\bar{V})$. If $r_n = 4^{-n-1}$, then the discs shrink at the same rate as the squares, and by producing arcs we can show $\liminf \gamma(D(z, \delta) \cap V)/\delta > 0$, which means that $\gamma(V) = \gamma(\bar{V})$. However if r_n is much smaller, for instance $r_n = 16^{-n-1}$, then we do not know whether $\gamma(V) = \gamma(\bar{V})$. We will show that $\alpha(\bar{V}) < 1.34\alpha(V)$ in this case.

Fix a, b with $0 < a < 1/4$, $0 < b < 1/4$. Let E be the Cantor set just described and let V be the open set corresponding to any choice of r_n such that $0 < r_n \leq ab^n$. Let f be a function C -admissible for the set \bar{V} . For any n , $f'(\infty)$ is the Cauchy integral of f around the boundaries of the sets $D_{m,k}, E_{n,j}$ where $1 \leq k \leq 4^m$, $0 \leq m \leq n-1$, $1 \leq j \leq 4^n$. We wish first to examine the integral around the $E_{n,j}$. We apply Theorem III.1.1 of [19] with $g =$ interior of $E_{n,j}$ and $e = E_{n,j}$. Let $\omega(\cdot)$ be the modulus of continuity of f , and let λ be the constant of Theorem III.1.1 for a square. (λ is an absolute constant; see the paragraph following the statement of Theorem III.1.1). Then

$$\left| \int_{\partial E_{n,j}} f(z) dz \right| \leq \lambda \cdot \omega(\alpha) \cdot \alpha$$

where

$$\alpha = \alpha(\text{interior } E_{n,j}) \leq \text{side of } E_{n,j} = 4^{-n}.$$

Therefore

$$\sum_{j=1}^{4^n} \left| \frac{1}{2\pi i} \int_{\partial E_{n,j}} f(z) dz \right| \leq \lambda \cdot \omega(4^{-n}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

From this we see that

$$\begin{aligned} f'(\infty) &= \sum_{m=0}^{n-1} \sum_{k=1}^{4^m} \frac{1}{2\pi i} \int_{\partial D_{m,k}} f(z) dz \\ &+ \sum_{j=1}^{4^n} \frac{1}{2\pi i} \int_{\partial E_{n,j}} f(z) dz \\ &\rightarrow \frac{1}{2\pi i} \int_S f(z) dz + 0. \end{aligned} \quad (11)$$

Now

$$\begin{aligned} \sum_{m=0}^{\infty} \sum_{k=1}^{4^m} \left| \frac{1}{2\pi i} \int_{\partial D_{m,k}} f(z) dz \right| &\leq \sum_{m=0}^{\infty} \sum_{k=1}^{4^m} (2\pi)^{-1} \cdot 1 \cdot (2\pi r_m) \\ &\leq \sum_{m=0}^{\infty} a(4b)^m = a/(1-4b). \end{aligned}$$

Thus

$$\alpha(\bar{V}) = \sup |f'(\infty)| = \sup \left| (2\pi i)^{-1} \int_S f(z) dz \right| \leq a/(1-4b).$$

But $\alpha(V) \geq \alpha(D_{0,1}) = r_0$ so

$$(12) \quad \alpha(\bar{V}) \leq \alpha(V) \cdot a/r_0 \cdot 1/(1-4b).$$

In the case that $r_n = 16^{-n-1}$, we have $a = b = r_0 = 1/16$, and $\alpha(\bar{V}) \leq 4/3 \cdot \alpha(V)$.

The inequality (12) is not very useful for b near $1/4$. Another way of comparing $\alpha(V)$ and $\alpha(\bar{V})$ is: $\alpha(\bar{V}) \leq \alpha(E_0) < 0.6$; $\alpha(V) \geq \alpha(D_{0,1}) = r_0$; and

$$(13) \quad \alpha(\bar{V}) < \alpha(V) \cdot 3/5r_0.$$

If it were possible to remove the requirement of continuity in (11), we would find that (12) holds with α replaced by γ . For let $L \subset \bar{V} - S$ be compact and let f be admissible for L . The integral of f on S makes sense because f is continuous on S . In fact f is analytic on a neighborhood of S . Then $|f'(\infty)| \leq a/(1-4b)$. Taking the supremum over f yields $\gamma(L) \leq a/(1-4b)$. Taking the supremum over L yields $\gamma(\bar{V}) = \gamma(\bar{V} - S) \leq a/(1-4b)$, according to Theorem 5, and so we proceed as before. Of course (13) does not require continuity of f on C , and so (13) does hold with γ replacing α .

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